Discipline: **Physics**
Subject: **Electromagnetic Theory**
Unit 25:
Lesson/ Module: **Energy Loss in Collisions - II**

Author (CW): **Prof. V. K. Gupta**
Department/ University: **Department of Physics and Astrophysics, University of Delhi, New Delhi-110007**

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Learning Objectives:

From this module students may get to know about the following:

1. Quantum effects in energy loss in collisions and the classical and quantum-mechanical energy loss formulas.

2. Density effects in energy loss due to polarization of the medium in dense media.

3. Modification in the energy loss formula in distant collisions due to the density effect.

25. Energy Loss in Collisions – II

25.1 Classical and quantum-mechanical energy loss formulas

In the last module we evaluated the energy loss by a particle in collision with free electrons on traversing through matter. We had also evaluated the energy transfer by a moving charged particle to a harmonically bound charge, a much better approximation to an electron bound in the nucleus. We can now use that result to calculate the energy loss per unit length by a charged particle moving through matter. We have a particle of charge *ze* and mass *M* passing by an electron (charge *e*, mass *m*) at an impact parameter *b* with velocity *v*. The electron is bound harmonically with a characteristic frequency ω_0 and a small damping constant Ѓ. The energy loss to such an electron by the moving charge is [see last module for details]

$$\Delta W(b) = (\frac{1}{4\pi\varepsilon_0})^2 \frac{2z^2e^4}{mv^2}(\frac{1}{b^2})[\xi^2 K_1{}^2(\xi) + \frac{1}{\gamma^2}\xi^2 K_0{}^2(\xi)] \tag{1}$$

Here

$$\xi = \frac{\omega_0 b}{\gamma v} = \frac{b}{b_{\max}}, \; b_{\max} = \frac{\gamma v}{\omega_0} \tag{2}$$

and K_0 and K_1 are modified Bessel functions of order zero and one, respectively. This formula is expected to be reasonable for all $b > b_{\min}$ [See module 24 for details.], where

$$b_{\min} = \frac{1}{4\pi\varepsilon_0}\frac{ze^2}{\gamma mv^2} \tag{3}$$

We suppose, as in module 24, that there are N atoms per unit volume of the matter through which the charged particle is passing, with Z electrons per atom. Of these Z electrons per atom, let f_j electrons have the characteristic frequency ω_j. The quantity f_j is called the *oscillator strength* of the oscillator with frequency ω_j. Obviously

$$\sum_j f_j = Z \tag{4}$$

The number of electrons having characteristic frequency ω_j and for which the impact parameter is between b and $b + db$ in a thickness dx of matter is

$$dn = Nf_j 2\pi b\, db\, dx$$

The energy loss to each one of them is given by equation (1) with ω_0 replaced by ω_j and consequently ξ by ξ_j:

$$\Delta W_j(b) = (\frac{1}{4\pi\varepsilon_0})^2 \frac{2z^2e^4}{mv^2}(\frac{1}{b^2})[\xi^2 K_1{}^2(\xi) + \frac{1}{\gamma^2}\xi^2 K_0{}^2(\xi)] \tag{5}$$

$$\xi_i = \frac{\omega_j b}{\gamma v} = \frac{b}{b_{\max}}, \; b_{\max} = \frac{\gamma v}{\omega_j} \tag{6}$$

Thus the total energy loss to all the electrons in the medium per unit length of travel of the incoming particle is

$$\frac{dW}{dx} = 2\pi N \sum_j f_j \int_{b_{\min}}^{\infty} \Delta W_j(b) b\, db \tag{7}$$

Now $b_{\min}$ given by equation (3) is the value of impact parameter below which formula given by equation (5) is not valid and the factor $\frac{1}{b^2}$ in equation (5) has to be replaced by $\frac{1}{b^2 + b_{\min}{}^2}$. As explained in module 24, this is effectively taken care of by restricting the integration range to $(b_{\min}, \infty)$ instead of $(0, \infty)$. On substituting for $\Delta W_j(b)$ from equation (5) and changing the variable of integration from b to $\xi = \frac{\omega_j b}{\gamma v}$, equation (7) takes the form

$$\frac{dW}{dx} = 2\pi N\left(\frac{1}{4\pi\varepsilon_0}\right)^2 \frac{2z^2e^4}{mv^2} \sum_j f_j \int_{\xi_{\min}}^{\infty} [\xi K_1{}^2(\xi) + \frac{1}{\gamma^2} \xi K_0{}^2(\xi)] d\xi \tag{8}$$

where

$$\xi_{\min} = \frac{\omega_j b_{\min}}{\gamma v} = \frac{ze^2\omega_j}{4\pi\varepsilon_0 m\gamma^2 v^3}. \tag{9}$$

The integral over ξ can be performed by using the well known recurrence relations for the Bessel functions:

$$K'_\nu(z) = -K_{\nu-1}(z) - \frac{\nu}{z} K_\nu(z)$$

$$K'_\nu(z) = -K_{\nu+1}(z) + \frac{\nu}{z} K_\nu(z)$$

From these two recurrence relations it follows that

$$z[K_0{}^2(z) + K_1{}^2(z)] = -\frac{d}{dz}[zK_0(z)K_1(z)]$$

$$-2zK_0{}^2(z) = \frac{d}{dz}[z^2\{K_1{}^2(z) - K_0{}^2(z)\}]$$

Now on substituting for K_0^2 and K_1^2 from these into equation (8) we obtain

$$\frac{dW}{dx} = 4\pi N\left(\frac{1}{4\pi\varepsilon_0}\right)^2 \frac{z^2e^4}{mv^2} \sum_j f_j [\xi_{\min} K_1(\xi_{\min}) K_0(\xi_{\min}) - \frac{v^2}{2c^2} \xi^2{}_{\min} \{K_1{}^2(\xi_{\min}) - K_0{}^2(\xi_{\min})\}] \tag{10}$$

On substituting the values of various factors in $\xi_{\min}$, we find that for relativistic particles $\xi_{\min} << 1$, so that the Bessel functions can be replaced by the leading term in their asymptotic expansion for small arguments:

$$K_0(z) \sim -\{\ln(0.5z) + 0.5772\}; K_1(z) \sim 1/z \qquad (11)$$

On using these asymptotic expressions in equation (11), we obtain

$$\frac{dW}{dx} = 4\pi NZ\left(\frac{1}{4\pi\varepsilon_0}\right)^2 \frac{z^2e^4}{mv^2}\left[\ln B_c - \frac{v^2}{2c^2}\right] \qquad (12)$$

where

$$B_c = \frac{1.123\gamma v}{<\omega> b_{\min}} = \frac{4\pi\varepsilon_0\gamma^2 mv^2}{ze^2 <\omega>} \qquad (13)$$

and

$$<\omega> = \exp\left[\frac{1}{Z}\sum_j f_j \ln \omega_j\right] \qquad (14)$$

is "average" characteristic frequency of the atom. This result was first obtained by Bohr in 1915. It differs from the approximate result that we derived earlier in module 24 [equation (15)] in the replacement of ω by $<\omega>$ and the additional term $\frac{v^2}{2c^2}$. This term is negligible at low energies and only a small correction even at high energies.

25.2 The quantum-mechanical effects

Bohr's formula (12) gives a reasonable description of the energy loss of heavier nuclei. However for lighter particles like electrons, mesons, protons and even high energy alpha particles, it overestimates the energy loss. For such lighter particles the quantum-mechanical effects become important and make considerable modification of the classical result. The two important quantum effects are due to (a) discreteness of the energy transfer from the incoming particle to the atom, and (b) wave nature of particles and uncertainty principle.

25.2.1 Discreteness of energy transfer

As we have seen in module 24, the energy transfer in a Coulomb collision is given by [see module 24 for details]

$$\Delta W(b) = \left(\frac{1}{4\pi\varepsilon_0}\right)^2 \frac{2z^2e^4}{mv^2}\frac{1}{b^2} \qquad (15)$$

The smallest energy transfer takes place for $b \cong b_{\max} = \frac{\gamma v}{\omega}$. From quantum theory, we know that energy transfer to the bound electron takes place in quanta. Now $\frac{e^2}{4\pi\varepsilon_0 \hbar c}$ is a dimensionless constant and numerically equals 1/137. So, numerically, $v_0 = \frac{e^2}{4\pi\varepsilon_0 \hbar} = \frac{c}{137}$. Similarly, $I_H = \frac{me^4}{2(4\pi\varepsilon_0)^2 \hbar^2}$ is the ionization potential of hydrogen atom, and numerically it equals 13.6 eV. In terms of these quantities the minimum energy transfer is given by

$$\Delta W(b) = \left(\frac{1}{4\pi\varepsilon_0}\right)^2 \frac{1}{\gamma^2} z^2 \left(\frac{v_o}{v}\right)^4 (\hbar\omega)\left(\frac{\hbar\omega}{I_H}\right) \tag{16}$$

Now $\left(\frac{\hbar\omega}{I_H}\right)$ is of the order of unity, and for fast moving relativistic particle, $v >> v_0 = c/137$, the energy loss $\Delta W(b)$ is much less than the ionization potential or even the excitation energy from the ground state. Since energy must be transferred in quanta equal to the excitation energy or ionization energy, in fact no energy transfer will take place. However in a certain statistical sense the classical formula is still valid. If we consider a large number of collisions, in most no energy will be transferred. But in a few collisions an appreciable amount of excitation energy is transferred. Thus on an average a tiny amount of energy, much smaller than the excitation energy, is indeed transferred. Thus in this statistical sense the classical formula is correct.

25.2.2 Wave nature of particles

The other important quantum modification comes from the wave nature of particles through uncertainty principle. If we try to construct a wave packet to give meaning to a classical trajectory then uncertainty principle tells us that the path can be defined to within uncertainty $\Delta x \geq \hbar / p$. If the impact parameter *b* is less than this uncertainty, the classical concept of a trajectory fails. Since the wave nature of the particles implies a smearing out of the trajectory for distances of the order of Δx, the correct quantum-mechanical energy loss will be much less than what is given by the classical result (see module 24). Thus $\Delta x \sim \hbar / p$ is the quantum analog of the minimum impact parameter, $b_{\min}$ of equation (3).

In the collision of two particles, each has a wave nature and is represented by a wave packet. For a given relative velocity the minimum uncertainty will come from the lighter of the two. For a heavy incident particle, and we are considering "heavy" incident particles, the momentum of the particle in the rest frame of the incident particle is $p' = \gamma m v$, where *m* is the mass of the electron. Thus the quantum-mechanical minimum impact parameter is

$$b_{\min}^{(q)} = \frac{\hbar}{\gamma m v}. \tag{17}$$

One limit on minimum value of impact parameter is due to the approximate formula used for the energy loss calculation and the other due to quantum effects. Thus in any situation, the larger of the two limits must be chosen. The ratio of the classical to quantum value of $b_{\min}$ is

$$\eta = \frac{ze^2}{4\pi\varepsilon_0 \hbar v} \tag{18}$$

If η>1, the classical Bohr formula is to be used. This is the case for slowly moving (v<<c) and highly charged (z>>1) incident particles. If η<1, the quantum mechanical limit on minimum impact parameter is larger than the classical one. In that case quantum modifications appear; in equation (16) of module 24 or equation (13) above, $b_{\min}$ must be replaced by $b_{\min}^{(q)}$. Thus

$$B^{(q)} = \frac{b_{\max}}{b_{\min}^{(q)}} = \frac{\gamma^2 m v^2}{\hbar\langle\omega\rangle} \tag{19}$$

This is a reasonably close approximation to the quantum-mechanical formula obtained by Bethe

$$\frac{dW^{(q)}}{dx} = 4\pi NZ \frac{1}{(4\pi\varepsilon_0)^2}\frac{z^2e^4}{mv^2}[\ln\frac{2\gamma^2 mv^2}{\hbar\langle\omega\rangle}) - (\frac{v}{c})^2] \tag{20}$$

The general behaviour of both the classical and quantum energy loss formulas, equations (12) and (20) respectively, is the same and is depicted in the figure below **[See Figure 13.4 from Jackson Edition 2]**.

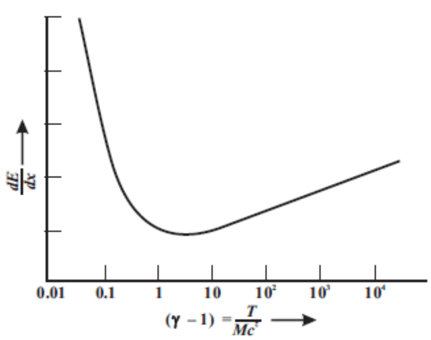

Fig: Energy loss as a function of kinetic energy

The energy loss dW/dx is plotted against the dimensionless quantity $(\gamma - 1 = \frac{T}{mc^2})$, where T is the kinetic energy of the particle, on a logarithmic scale. At low energies the variation is mainly due to the factor $1/v^2$ which varies much faster than the logarithmic term and the energy loss falls off as $1/v^2$. But at very high energies, $v \to c$ and hence its variation is negligible. The factor γ in the logarithm now becomes dominant, since $\gamma \to \infty$ as $v \to c$. As a result the energy loss begins to increase again, albeit slowly. The Bethe energy loss formula (20) is in good agreement with experiment for all fast particles with η<1, provided the energy of the particle is not too high in which case density effects of the matter become important and lead to certain modifications in the formula.

It is of interest to note the physical origins of the two powers of γ that appear in B_c [equation (13)] or $B^{(q)}$ [equation (19)]. One factor of γ comes from the increase of the maximum energy that can be transferred in a head-on collision. [See module 24 for details.] As a result, $b_{\min}$ or $b_{\min}^{(q)}$ is proportional to γ^{-1}. The other factor of γ comes from the relativistic effect on the electromagnetic field. These fields were obtained in detail in module 16 and are given by equation (1) of module 24. As discussed in module 16, the time interval over the fields are appreciable over times $\sim \frac{b}{\gamma v}$. This makes $b_{\max}$ [equation (6)] proportional to γ. In other words, the fields are effective in transferring energy up to alarger distance in a relativistic particle than in a non-relativistic particle. B_c and $B^{(q)}$ being essentially ratio of $b_{\max}$ to $b_{\min}$, are thus proportional to γ^2.

25.3 Density effects in energy loss

For particles which are relativistic but not ultra relativistic, the observed energy loss is quite well represented by equation (20) when quantum effects dominate or else by equation (12). For ultra relativistic particles, however, the observed energy loss is less than what is predicted by equation (20), especially for dense substances. In terms of the figure above, the energy loss increases beyond the minimum but at a much slower rate; the slope of the curve is nearly half of what is depicted in the figure. This implies that the energy loss increases *not as log of* γ^2 *but as log of* γ.

This reduction in the rate of energy loss is known as *density effect*. It was first treated by Enrico Fermi in 1940. In our discussion so far, we have made one tacit assumption that is actually not valid in dense materials. We have calculated the effect of the field of the incoming particle on one electron at a time, calculated the energy loss and then added up incoherently the energy transfer to all the electrons in all the atoms with impact parameter lying between $b_{\min}$ and $b_{\max}$. Now $b_{\max} = \frac{\gamma v}{\omega}$ increases with γ and for large γ can become quite large compared to atomic dimensions. As a result, there are many atoms lying in the trajectory of the incident particle having impact parameter up to $b_{\max}$. These atoms are all influenced by the field of the incoming particle. Choosing one of the atoms for consideration, all these atoms that lie within $b_{\max}$ of this atom will affect with the perturbing fields of their own. The same thing can be explained in the language of *polarization* of the medium. In a dense medium, the dielectric polarization of the material changes the field of the incoming particle from its free-space value to that characteristic of the macroscopic field in a dielectric material. Obviously, as a result, energy transfer calculations will also be modified.

25.3.1 Energy loss in distant collisions

We will determine the energy loss in distant collisions ($b>a$, the atomic dimension), assuming that the fields in the medium can be calculated in the continuum approximation of a macroscopic dielectric constant $\kappa(\omega)$ or permittivity $\varepsilon(\omega)$.

Maxwell's equations for a linear material were obtained in module 3 and are

$$\vec{\nabla}.\vec{E} = \rho/\varepsilon\,, \qquad \vec{\nabla}\times\vec{B} = \mu(\vec{J} + \varepsilon\partial\vec{E}/\partial t)$$

$$\vec{\nabla}\times\vec{E} + \frac{\partial\vec{B}}{\partial t} = 0\,, \qquad \vec{\nabla}.\vec{B} = 0$$

The charge and current densities are the free charge and current densities here. The equations appear identical to the ones in free space except for the replacement $\varepsilon_0 \to \varepsilon$, and $\mu_0 \to \mu$. For a non-magnetic material, we can take $\mu = \mu_0$. On writing $\varepsilon = \varepsilon_0\kappa$, where κ is the dielectric constant of the medium, and $\mu_0\varepsilon_0 = 1/c^2$, the equations for the scalar and vector potentials in the Lorentz gauge are

$$\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = -\rho/(\varepsilon_0\kappa) \tag{21}$$

$$\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = -\mu_0\vec{J} \tag{22}$$

If we now take the Fourier transforms of the potentials $(\Phi, \vec{A})$ and the sources $(\rho, \vec{J})$

$$f(\vec{x},t) = \frac{1}{(2\pi)^2}\int d^3k \int d\omega f(\vec{k},\omega) e^{i\vec{k}.\vec{x} - i\omega t} \tag{23}$$

where f refers to any one of the four quantities. When we take the Fourier transforms, the operator $\frac{\partial}{\partial t}$ is replaced by $(-i\omega)$ and $\vec{\nabla}$ by $(i\vec{k})$. On taking the Fourier transforms of equation (23) and using these replacements, the wave equations for Φ and $\vec{A}$ become

$$\left[k^2 - \frac{\omega^2}{c^2}\kappa(\omega)\right]\Phi(\vec{k},\omega) = \frac{\rho(\vec{k},\omega)}{\varepsilon_0\kappa(\omega)} \tag{24}$$

$$\left[k^2 - \frac{\omega^2}{c^2}\kappa(\omega)\right]\vec{A}(\vec{k},\omega) = \mu_0\vec{J}(\vec{k},\omega) \tag{25}$$

For a charged particle of charge *ze* moving with velocity $\vec{v}$, we have

$$\rho(\vec{x},t) = ze\delta(\vec{x} - \vec{v}t) \tag{26}$$

$$\vec{J}(\vec{x},t) = \vec{v}\rho(\vec{x},t) \tag{27}$$

The Fourier transforms of these expressions are

$$\rho(\vec{k},\omega) = \frac{1}{(2\pi)^2}\int d^3x\int dt\rho(\vec{x},t)e^{-(i\vec{k}.\vec{x}-i\omega t)} = \frac{ze}{(2\pi)^2}\int d^3x\int dt\delta(\vec{x}-\vec{v}t)e^{-(i\vec{k}.\vec{x}-i\omega t)}$$
$$= \frac{ze}{(2\pi)^2}\int dt e^{i[\omega-\vec{k}.\vec{v}]t} = \frac{ze}{(2\pi)}\delta(\omega-\vec{k}.\vec{v}) \tag{28}$$

Similarly

$$\vec{J}(\vec{k},\omega) = \vec{v}\rho(\vec{k},\omega) \tag{29}$$

On substituting equations (28) and (29) into equations (24) and (25) respectively, we get

$$\Phi(\vec{k},\omega) = \frac{ze}{2\pi\varepsilon_0\kappa(\omega)}\frac{\delta(\omega-\vec{k}.\vec{v})}{k^2-\frac{\omega^2}{c^2}\kappa(\omega)} \tag{30}$$

$$\vec{A}(\vec{k},\omega) = \frac{\mu_0 ze}{2\pi}\vec{v}\frac{\delta(\omega-\vec{k}.\vec{v})}{k^2-\frac{\omega^2}{c^2}\kappa(\omega)} = \mu_0\varepsilon_0\kappa(\omega)\vec{v}\Phi(\vec{k},\omega) = \frac{\kappa(\omega)\vec{v}}{c^2}\Phi(\vec{k},\omega) \tag{31}$$

The Fourier transforms of the electromagnetic fields are obtained from the potentials by using their definitions:

$$\vec{E} = -\vec{\nabla}\Phi - \frac{\partial\vec{A}}{\partial t}; \qquad \vec{B} = \vec{\nabla}\times\vec{A} \tag{32}$$

Since on taking Fourier transform, the operator $\frac{\partial}{\partial t}$ is replaced by $(-i\omega)$ and $\vec{\nabla}$ by $(i\vec{k})$,

$$\vec{E}(\vec{k},\omega) = i[\frac{\omega\kappa(\omega)\vec{v}}{c^2}-\vec{k})]\Phi(\vec{k},\omega) \tag{33}$$

$$\vec{B}(\vec{k},\omega) = \frac{i\kappa(\omega)}{c^2}(\vec{k}\times\vec{v})\Phi(\vec{k},\omega) \tag{34}$$

Now the energy loss formula was obtained in the last module [equation (28) of the last module], and is

$$\Delta W = 2e\,\mathrm{Re}\int_0^\infty (i\omega)\vec{x}(\omega).\vec{E}^*(\omega)d\omega \tag{35}$$

To find $\vec{E}(\omega)$ we need the inverse Fourier transform of $\vec{E}(\vec{k},\omega)$:

$$\vec{E}(\omega) = \frac{1}{(2\pi)^{3/2}}\int d^3k\vec{E}(\vec{k},\omega)e^{i\vec{k}.\vec{x}} \tag{36}$$

We take the particle to be moving along the *x*-axis and the observation point, at a distance *b* along the *y*-axis, has the coordinates (*0, b, 0*). So

$$\vec{E}(\omega) = \frac{1}{(2\pi)^{3/2}} \int d^3k \vec{E}(\vec{k},\omega) e^{ibk_y} \tag{37}$$

Let us find the *x*- component of the field, the component along the direction of velocity of the moving charge. On using equation (30), (31) and (33) we obtain

$$E_x(\omega) = \frac{1}{(2\pi)^{3/2}} \frac{ize}{2\pi\varepsilon_0 \kappa(\omega)} \int d^3k e^{ibk_y} [\frac{\omega\kappa(\omega)v}{c^2} - k_x] \frac{\delta(\omega - vk_x)}{k^2 - \frac{\omega^2}{c^2}\kappa(\omega)} \tag{38}$$

The integral over k_x can be done trivially because of the delta function. The result is

$$E_x(\omega) = \frac{1}{(2\pi)^{3/2}} \frac{ize\omega}{2\pi\varepsilon_0 v^2} [\frac{v^2}{c^2} - \frac{1}{\kappa(\omega)}] \int dk_y e^{ibk_y} \int dk_z \frac{1}{k_y{}^2 + k_z{}^2 + \lambda^2} \tag{39}$$

where

$$\lambda^2 = \frac{\omega^2}{v^2} - \frac{\omega^2}{c^2}\kappa(\omega) = \frac{\omega^2}{v^2}[1 - \beta^2 \kappa(\omega)] \tag{40}$$

The integral over k_z is elementary:

$$\int_{-\infty}^{\infty} \frac{dk_z}{k_z{}^2 + k_y{}^2 + \lambda^2} = \frac{\pi}{[k_y{}^2 + \lambda^2]^{1/2}}) \tag{41}$$

On inserting equation (41) into equation (39) we obtain

$$\begin{aligned} E_x(\omega) &= \frac{1}{(2\pi)^{3/2}} \frac{ize\omega}{2\pi\varepsilon_0 v^2} [\beta^2 - \frac{1}{\kappa(\omega)}] \int dk_y e^{ibk_y} \frac{\pi}{[k_y{}^2 + \lambda^2]^{1/2}} \\ &= \frac{1}{(2\pi)^{1/2}} \frac{ize\omega}{4\pi\varepsilon_0 v^2} [\beta^2 - \frac{1}{\kappa(\omega)}] \int dk_y e^{ibk_y} \frac{1}{[k_y{}^2 + \lambda^2]^{1/2}} \end{aligned} \tag{42}$$

The same integral we had encountered in the last module as well while considering energy transfer to a harmonically bound charge. It is expressed in terms of modified Bessel function of order unity:

$$K_0(z) = \frac{1}{2} \int_{-\infty}^{\infty} \frac{e^{izt} dt}{(1+t^2)^{\frac{1}{2}}} \tag{43}$$

Using this form in equation (42) we obtain

$$E_x(\omega) = -(\frac{2}{\pi})^{1/2} \frac{ize\omega}{4\pi\varepsilon_0 v^2}[\frac{1}{\kappa(\omega)} - \beta^2]K_0(\lambda b) \tag{44}$$

An exactly similar calculation will yield E_y:

$$E_y(\omega) = (\frac{2}{\pi})^{1/2} \frac{ze}{4\pi\varepsilon_0 v} \frac{\lambda}{\kappa(\omega)} K_1(\lambda b) \tag{45}$$

From equations (33) and (34) we find

$$B_z = \frac{\kappa(\omega)v}{c^2} E_y \tag{46}$$

As expected, in the limit of $\kappa(\omega) \to 1$, equations (44) and (45) reduce to the corresponding equations [equations (38) and (37)] of the last module, respectively.

The energy transfer can now be obtained from a generalization of equation (35)

$$\Delta W = 2e\sum_j f_j \operatorname{Re}\int_0^\infty (i\omega)\vec{x}_j(\omega).\vec{E}^*(\omega)d\omega \tag{47}$$

Here $\vec{x}_j(\omega)$ is the amplitude of the jth type of electron in the atom. The expression for $\vec{x}_j(\omega)$ was derived in the last module, and is

$$\vec{x}(\omega_j) = -\frac{e}{m}\frac{\vec{E}(\omega)}{\omega_j{}^2 - i\omega\Gamma - \omega^2} \tag{48}$$

Further, on expressing $\sum_j \frac{f_j}{\omega_j{}^2 - \omega^2 - i\omega\gamma_j}$ in terms of the dielectric constant of the medium **[see module on electromagnetic waves for details]**

$$\kappa(\omega) = 1 + \frac{Ne^2}{\varepsilon_0 m}\sum_j \frac{f_j}{\omega_j{}^2 - \omega^2 - i\omega\gamma_j} \tag{49}$$

we obtain

$$\Delta W(b) = \frac{2\varepsilon_0}{N} \operatorname{Re}\int_0^\infty [i\omega\kappa(\omega)]\left|\vec{E}(\omega)\right|^2 d\omega \tag{50}$$

Here N is the number of atoms per unit volume. The energy loss per unit distance in collisions wherein the impact parameter b is greater than or equal to a is

$$\left(\frac{dW}{dx}\right)_{b\geq a} = 2\pi N\int_a^\infty \Delta W(b)bdb \tag{51}$$

On using equations (44) and (45) for the field $\vec{E}(\omega)$, we obtain

$$\left(\frac{dW}{dx}\right)_{b\geq a} = \frac{2}{\pi}\frac{z^2e^2}{4\pi\varepsilon_0}\frac{1}{v^2}\operatorname{Re}\int_0^\infty i\omega(\lambda^* a)[\frac{1}{\kappa(\omega)} - \beta^2]K_0(\lambda a)K_1(\lambda^* a)d\omega$$

Alternatively, to find the energy lost due to collisions with impact parameter greater than or equal to *a*, we calculate the radially outward component of the Poynting vector, $\vec{S}$. When this component is integrated over all time and over a closed loop of radius *a*, we get the total field energy which flows away from the particle per unit length of the path. From the law of conservation of energy, this is the energy lost by the incident particle. Thus

$$\left(\frac{dW}{dx}\right)_{b>a} = \int_{-\infty}^\infty 2\pi a(\vec{S}.\hat{n})dt = -\frac{2\pi a}{\mu_0}\int_{-\infty}^\infty B_z(t)E_x(t)dt \tag{52}$$

On introducing the Fourier transforms of $B_z(t)$ and $E_x(t)$ in the standard way, we obtain

$$\left(\frac{dW}{dx}\right)_{b>a} = -\frac{4\pi a}{\mu_0}\operatorname{Re}\int_0^\infty B_z^*(\omega)E_x(\omega)d\omega \tag{53}$$

Now when we substitute for $B_z^*(\omega)$ and $E_x(\omega)$ from equations (44), (45) and (46), we obtain

$$\left(\frac{dW}{dx}\right)_{b\geq a} = \frac{2}{\pi}\frac{z^2e^2}{4\pi\varepsilon_0}\frac{1}{v^2}\operatorname{Re}\int_0^\infty i\omega(\lambda^* a)[\frac{1}{\kappa(\omega)} - \beta^2]K_0(\lambda a)K_1(\lambda^* a)d\omega \tag{54}$$

This is the expression for the Energy loss first obtained by Fermi. This expression may appear to be very different from the result that we obtained earlier [equation (10)]. But in the limit of polarization effects being unimportant, it reduces to the same result.

In equation (54), the argument of the modified Bessel functions is in general complex, since $\kappa(\omega)$ is in general complex. This is the origin of the density effect. From equation (40) we see that $\kappa(\omega)$ is multiplied by the factor β^2 and so the density effects are really important for very high energies, so that $\beta \simeq 1$.

Summary:

1. *In this module the energy loss per unit length by a charged particle moving through matter is calculated.*

2. *First the classical formula for energy loss due to Bohr is derived.*

3. *Quantum mechanical modifications to the formula are discussed.*

4. *It is explained how the energy transfer rapidly goes to zero for impact parameter beyond certain maximum due the discrete nature of the energy transfer.*

5. *Secondly, due to the uncertainty relation the classical trajectory is not defined for distances,* $\Delta x \geq \hbar / p$ *giving* $\Delta x \sim \hbar / p$ *as the quantum analog of the classical minimum impact parameter. It is explained that for "slow" moving particles the classical formula is to be used, whereas for "high" energy particles the quantum minimumimpact parameter is to be used.*

6. *The effect of the density of the medium, particularly for ultra-relativistic particles is discussed. Formula for the energy loss including the effect of the density via the polarization of the medium is obtained.*